APPENDIX III

AN ALGORITHM FOR LEAST-SQUARES RATIONAL APPROXIMATION OF COMPLEX FUNCTIONS ON THE UNIT CIRCLE

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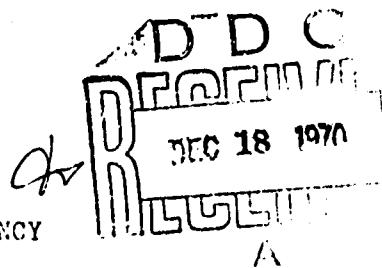University of Uppsala

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FUNCTIONS ON THE UNIT CIRCLE

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Abstract. An algorithm for rational approximation on the unit circle will be given. An immediate application is computation of recursive filters for use on a digital computer.

1. Definitions

Let $f(\omega)$ be a given complex-valued function defined for real values of ω, and let

$$R_{m,n}(\omega) = \frac{a_0 + a_1 e^{-i\omega} + \ldots + a_m e^{-im\omega}}{b_0 + b_1 e^{-i\omega} + \ldots + b_n e^{-in\omega}} \tag{1}$$

be a complex-valued rational approximating function defined for real values of ω. The sets $A = \{a_0, a_1, \ldots, a_m\}$ and $B = \{b_0, b_1, \ldots, b_n\}$ are real parameters of $R_{m,n}(\omega)$. In the formulas below a_i and b_i are assumed to be equal to zero if they do not belong to A or B, i.e. if their subscripts are outside the intervals $0 - m$ or $0 - n$ respectively. We will always assume that $m \geq n$.

2. Least-squares approximation

We want to determine the parameters $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_n$ such that

$$L_2(f - R) = \left[\int_0^{2\pi} \left| f(\omega) - R_{m,n}(\omega) \right|^2 d\omega \right]^{1/2} \tag{2}$$

attains a minimum (assuming it exists). The "linear" approach of taking
the partial derivatives with respect to all the parameters is not
feasible, as it will result in a system of non-linear equations which
is hard to solve. We will avoid this difficulty by using an algorithm
given by the author (Jansson, 1965).

3. The algorithm

Following the algorithm mentioned in the last paragraph we start
by adding a real constant h to $f(\omega)$, thus having $g(\omega) = f(\omega) + h$.
Let $B^0 = \{1, 0, \ldots, 0, \ldots, 0\}$ and consider $L_2(g - R)$ as a function
of $a_0, a_1, \ldots, a_n$ only. We can minimize $L_2(g - R)$ by minimizing
$L_2(g - R)^2$, and this makes the analysis simpler. It may be shown that
a unique minimum exists. Furthermore, it is clear that $L_2(g - R)$ does
not have a maximum value, for we can place $R(\omega)$ as far away from $f(x)$
as we please. Since $L_2(g - R)^2$ is a differentiable function of
$a_0, a_1, \ldots, a_n$, the minimum must occur where all of the partial
derivatives

$$\frac{\partial L_2(g - R)^2}{\partial a_k}, \qquad\qquad k = 1, 2, \ldots, n, \qquad (3)$$

are zero.

From the fact that $L_2(g - R)$ is a strictly convex function of
$a_0, a_1, \ldots, a_m$, it follows that there is only one set of parameters
for which all of the partial derivatives are zero.

We have

$$I = |g(\omega) - R_{m,n}(\omega)|^2 =$$

$$= |g(\omega)|^2 + |R_{m,n}(\omega)|^2 - Re\{g(\omega)\, R_{m,n}(\omega)\}. \qquad (4)$$

where

$$|R_{m,n}(\omega)|^2 =$$

$$= \frac{(a_0^2 + a_1^2 + \ldots + a_n^2) + 2(a_0 a_1 + a_1 a_2 + \ldots + a_{m-1} a_m) \cos \omega + \ldots + 2(a_0 a_m) \cos m\omega}{(b_0^2 + b_1^2 + \ldots + b_n^2) + 2(b_0 b_1 + b_1 b_2 + \ldots + b_{n-1} b_n) \cos \omega + \ldots + 2(b_0 b_n) \cos n\omega}$$

and

$$Re\{g(\omega) R_{m,n}(\omega)\} =$$

$$= \left[Re\{g(\omega)\} Re\{P_m(\omega) \overline{Q_n(\omega)}\} - \right.$$

$$\left. - Im\{g(\omega)\} Im\{P_m(\omega) \overline{Q(\omega)}\} \right] / |Q_n(\omega)|^2.$$

We have that

$$P_m(\omega) \overline{Q_n(\omega)} = a_0 b_n e^{in\omega} + a_0 b_{n-1} e^{i(n-1)\omega} + \ldots +$$

$$+ (a_0 b_0 + a_1 b_1 + \ldots + a_n b_n) + (a_1 b_0 + a_2 b_1 + \ldots + a_{n+1} b_n) e^{-i\omega} +$$

$$+ \ldots + a_m b_0 e^{-im\omega}. \tag{5}$$

It is now easy to compute the partial derivatives

$$\frac{\partial I}{\partial a_k} = 2 \left[\sum_{j=0}^{m} (a_{k+j} + a_{k-j}) \cos j\omega - \right.$$

$$\left. - Re\{g(\omega)\} \sum_{j=-m}^{m} b_{k+j} \cos j\omega + Im\{g(\omega)\} \sum_{j=-m}^{m} b_{k+j} \sin j\omega \right] /$$

$$/ |Q_n(\omega)|^2 \tag{6}$$

Setting all the partial derivatives equal to zero and substituting
into (3) we get the matrix equation

$$M_A \cdot A^0 = G_A \tag{7}$$

M_A is a $m \times m$ Toeplitz matrix, A^0 is the column vector
$(a_0, a_1, \ldots, a_m)^T$, and G_{Ai} a column vector. Let us write out the
elements of M_A and G_A

$$M_{Aij} = \int_0^{2\pi} \frac{\cos |i - j| \omega}{|Q_n(\omega)|^2} \, d\omega$$

$$G_{Ai} = \int_0^{2\pi} \left[\text{Re}\{g(\omega)\} \sum_{j=-m}^{m} b_{j+i} \cos j\,\omega - \right.$$
$$\left. - \text{Im}\,\{g(\omega)\} \sum_{j=-m}^{m} b_{j+i} \sin j\,\omega \right] \cdot |Q_n(\omega)|^{-2} \, d\omega \tag{8}$$

To solve for A we just have to compute

$$A^0 = M_A^{-1} G_A$$

which is very easy since M is a Toeplitz matrix that can be inverted
in cm^2 operations. Here c is a constant and m is the number of rows
and columns. An algorithm for this purpose has been given by
Robinson (1963).

We have now done the first step and computed a trial solution A^0.
Let us in the next step compute a trial solution B^1. We will do this by
minimizing

$$L_2\left(\cdot^2\left(\frac{1}{R} - \frac{1}{\hat{r}}\right)\right) = \left[\int_0^{2\pi} \left|g^2(\omega)\left(\frac{1}{R_{m,n}(\omega)} - \frac{1}{\hat{g}(\omega)}\right)\right|^2 d\omega\right]^{1/2} \tag{9}$$

Let us write

$$I = \left|g^2(\omega)\left(\frac{1}{R_{m,n}(\omega)} - \frac{1}{\hat{r}(\omega)}\right)\right|^2 =$$

$$= \left|g^2(\omega)\left(\frac{Q_n(\omega)}{P_m(\omega)} - \frac{1}{g(\omega)}\right)\right|^2 =$$

$$= \left|g^2(\omega)\frac{Q_n(\omega)}{P_m(\omega)}\right|^2 - |g(\omega)|^2 - 2\text{Re}\left\{g^2(\omega)\frac{Q_n(\omega)}{P_m(\omega)} - g(\omega)\right\} \tag{10}$$

The parameters B enter linearly in the expression above so that we may take the partial derivatives and solve for B^1 just as we solved for A^0.

Again we get a matrix equation

$$M_B B^1 = G_B \tag{11}$$

where M_B is a Toeplitz matrix and G_B is a column vector. We solve for B^1 by computing

$$B^1 = M_B^{-1} G_B.$$

The elements of M_B and G_B may be written

$$M_{Bij} = \int_0^{2\pi} \frac{g^2(\omega)\cos|i-j|\omega}{|P_m(\omega)|^2} d\omega$$

$$
G_{Bij} = \int_0^{} \left[\text{Im} \{ r^{2}(\omega) \} \sum_{j=-m}^{m} n_{j+i} \cos j\omega - \right.
$$

$$
\left. - \text{Im} \{ r^{2}(\omega) \sum_{j=-m}^{m} n_{j+i} \sin j\omega \right] \cdot |t_{\pi}(\omega)|^{-2} d\omega \qquad (12)
$$

We have now completed one iteration of the algorithm and we just have
to continue to determine A^s and B^s, A^{s+1} and L^{s+1} etc until we are
close enough to the point in the parameter space AUB where the
minimum (if it exists) is located. We have then computed a sufficiently
good approximation $R_{m,n}^{G}(\omega)$ to $g(\omega)$. Obviously

$$
R_{m,n}^{f}(\omega) = R_{m,n}^{G}(\omega) - h = \frac{P_m^{G}(\omega) - h\, Q_n^{G}(\omega)}{Q_n^{G}(\omega)} = \frac{P_m^{f}(\omega)}{Q_n^{G}(\omega)} \qquad (13)
$$

is an equally good approximation to $f(\omega)$.

References

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